

Born-Infeld black holes in the presence of a cosmological constant

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Abstract

We construct asymptotically anti-deSitter (and deSitter) black hole solutions of Einstein-Born-Infeld theory in arbitrary dimension. We critically analyse their geometries and discuss their thermodynamic properties.

1 Introduction

By now, there are substantial evidences which suggests that the type IIB string theory on $AdS_5 \times S_5$ is dual to four dimensional $N = 4$ super Yang-Mills theory [1]. At high temperature, the thermal state of the string theory is described by an asymptotically AdS black hole. Therefore, qualitatively the properties of thermal super Yang-Mills can be understood by studying the black hole geometry. However, due to the fact that the bulk supergravity describes gauge theory at strong coupling, the quantitative understanding becomes difficult. Nevertheless, many attempts were made along this direction (see for example [2], [3], [4], [5], [6]). By going higher in the bulk coupling, we make the dual boundary theory weaker. For example, one may wonder if we can atleast qualitatively understand the behaviour of the boundary Yang-Mills theory as we perturb the black hole geometry by turning on the higher derivative curvature terms in the bulk action. In fact, such an analysis was carried out in [7] by constructing bulk anti-deSitter black holes in the presence of certain higher curvature terms in the supergravity action. Furthermore, black holes of higher curvature gravity were constructed where the electromagnetic coupling was turned on [8] *. In general, beside the curvature terms, one would also expect higher derivative gauge field contributions to the supergravity actions. How does the boundary theory respond when we incorporate such corrections? As a first step to analyse such an issue, in this paper, we study the effect of adding higher derivative gauge field terms on the bulk adS (dS) black hole geometry. This is done by explicitly constructing black hole solutions of the supergravity action coupled to a Born-Infeld gauge field in arbitrary dimension in the presence of a cosmological constant. This action not only incorporates the higher order gauge field corrections to the Einstein-Maxwell gravity in the presence of a cosmological constant, but also allows us to find exact black hole solutions. Note that in general the gravity action may have both higher order curvature terms as well as the higher derivative terms due to gauge fields. We have not analysed here the black holes in these.

In the remaining part of the paper, we first construct black holes in $(n + 1)$ dimensional Einstein-Born-Infeld theory in the presence of a negative or positive cosmological constant[†]. Next, we study the thermodynamics of these holes. Here, in particular, we check that these black holes follow first law of thermodynamics. Then, by calculating specific heat at fixed charge, we show that for a certain range of parameters these black holes are stable. Expression of free energy at fixed charge is also calculated for these holes. Furthermore, by expanding our solutions around Reissner-Nordstrom anti deSitter (RNadS) black holes [9], we find out the effect of higher order gauge field corrections to the geometry.

*Under IIB string compactification, the electromagnetic field appear when we take the compact space to be a spinning sphere.

[†]In three and four dimensions the solutions were constructed in [10] and [11] respectively. However, their thermodynamical behaviour was not analysed.

2 AdS and dS Black hole solutions of Einstein-Born-Infeld theory

In this section, we will explicitly construct the black hole solutions of Einstein-Born-Infeld action in $(n + 1)$ dimension in the presence of a negative (or positive) cosmological constant Λ . The action has the form

$$S = \int d^{n+1}x \sqrt{-g} \left[\frac{(R - 2\Lambda)}{16\pi G} + L(F) \right], \quad (1)$$

where $L(F)$ is given by

$$L(F) = 4\beta^2 \left(1 - \sqrt{1 + \frac{F^{\mu\nu} F_{\mu\nu}}{2\beta^2}} \right). \quad (2)$$

The constant β is the Born-Infeld parameter and has the dimension of mass. In the limit $\beta \rightarrow \infty$ $L(F)$ reduces to the standard Maxwell form

$$L(F) = -F^{\mu\nu} F_{\mu\nu} + \mathcal{O}(F^4). \quad (3)$$

The black holes in this limit were constructed in [9] and their thermodynamics and phase structures were studied in [13], [9]. For simplicity, in this paper, we will work with the convention that $16\pi G = 1$, where G is the Newton's constant.

By varying the action with respect to the gauge field A_μ and the metric $g_{\mu\nu}$, we get the corresponding equations of motion. These are respectively

$$\nabla_\mu \left(\frac{F^{\mu\nu}}{\sqrt{1 + \frac{F^2}{2\beta^2}}} \right) = 0, \quad (4)$$

and

$$R_{\mu\nu} + \frac{2}{n-1} g_{\mu\nu} \Lambda = \frac{1}{n-1} g_{\mu\nu} L + \frac{2}{n-1} \frac{g_{\mu\nu} F^2}{\sqrt{1 + \frac{F^2}{2\beta^2}}} - \frac{2F_{\alpha\mu} F^\alpha{}_\nu}{\sqrt{1 + \frac{F^2}{2\beta^2}}}. \quad (5)$$

In order to solve the equations of motion, we use the metric ansatz

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{n-1}^2, \quad (6)$$

where, $d\Omega_{n-1}^2$ denotes the metric of a unit $(n - 1)$ sphere. $V(r)$ is an unknown function of r which we will determine shortly. First of all, a class of solution of equation (4) can immediately be written down where all the components of $F^{\mu\nu}$ are zero except F^{rt} . It is given by

$$F^{rt} = \frac{\sqrt{(n-1)(n-2)}\beta q}{\sqrt{2\beta^2 r^{2n-2} + (n-1)(n-2)q^2}}. \quad (7)$$

Here q is an integration constant and is related to the electromagnetic charge. This can be concluded from the behaviour of F^{rt} in the large β limit as $F^{rt} \sim \frac{q}{r^{n-1}}$. We notice that the electric field is finite at $r = 0$. This is expected in Born-Infeld theories. Now, parametrising $\Lambda = -\frac{n(n-1)}{2l^2}$, equation (6) can easily be solved as

$$V(r) = 1 - \frac{m}{r^{n-2}} + \left[\frac{4\beta^2}{n(n-1)} + \frac{1}{l^2} \right] r^2 - \frac{2\sqrt{2}\beta}{(n-1)r^{n-2}} \int \sqrt{2\beta^2 r^{2n-2} + (n-1)(n-2)q^2} dr. \quad (8)$$

The integral can be done in terms of hypergeometric function and can be written in a compact form. The result is

$$V(r) = 1 - \frac{m}{r^{n-2}} + \left[\frac{4\beta^2}{n(n-1)} + \frac{1}{l^2} \right] r^2 - \frac{2\sqrt{2}\beta}{n(n-1)r^{n-3}} \sqrt{2\beta^2 r^{2n-2} + (n-1)(n-2)q^2} \\ + \frac{2(n-1)q^2}{nr^{2n-4}} {}_2F_1 \left[\frac{n-2}{2n-2}, \frac{1}{2}, \frac{3n-4}{2n-2}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r^{2n-2}} \right]. \quad (9)$$

In the above expression, m appears as an integration constant and is related to the ADM mass of the configuration. It can be checked that for $n = 3$, it reduces to the solution of [11]. Also, in the $l \rightarrow \infty$ limit (that is for $\Lambda = 0$), this solution reproduces correctly the asymptotically flat Born-Infeld black hole (see for example [12]).

Using the fact that ${}_2F_1(a, b, c, z)$ has a convergent series expansion for $|z| < 1$, we can find the behaviour of the metric for large r . This is given by

$$V(r) = 1 - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2n-4}} + \frac{r^2}{l^2} - \frac{(n-1)(n-2)^2 q^4}{8\beta^2(3n-4)r^{4n-6}}, \quad (10)$$

Note that in the $\beta \rightarrow \infty$, it has the form of Reissner-Nordstrom adS black hole [9]. The last term in the right hand side of the above expression is the leading Born-Infeld correction to the RNadS black hole in the large β limit. From the asymptotic behaviour, we see that m is related to the mass of the configuration. In particular, in our convention, the ADM mass M is

$$M = (n-1)\omega_{n-1}m, \quad (11)$$

where ω_{n-1} is the volume of the unit $(n-1)$ sphere. More interesting is the behaviour of $V(r)$ close to the origin where

$$V(r) = 1 - \frac{m-A}{r^{n-2}} - \left[\frac{2c\beta}{n} - B(2n-1)q \right] \frac{q}{r^{n-3}} + \left[\frac{4\beta^2}{n(n-1)} + \frac{1}{l^2} \right] r^2 - \left[\frac{2c\beta}{n} + B \right] \frac{\beta^2 r^{n+1}}{(n-1)(n-2)}, \quad (12)$$

where

$$A = \frac{2(n-1)q^2}{n\sqrt{\pi}} \left\{ \frac{2\beta^2}{(n-1)(n-2)q^2} \right\}^{\frac{n-2}{2n-2}} \Gamma \left[\frac{3n-4}{2n-2} \right] \Gamma \left[\frac{1}{2n-2} \right], \quad (13)$$

and

$$c = \sqrt{\frac{2(n-2)}{(n-1)}} \quad \text{and} \quad B = \frac{4\beta}{cn(2n-1)q} \frac{\Gamma \left[\frac{3n-4}{2n-2} \right] \Gamma \left[\frac{-1}{2n-2} \right]}{\Gamma \left[\frac{n-2}{2n-2} \right] \Gamma \left[\frac{2n-3}{2n-2} \right]}.$$

From the above expression, we see that for generic values of $n \geq 3$, the metric has a curvature singularity at $r = 0$. However, for $n = 3$ and for $m = A$, the metric is regular at $r = 0$. We now proceed to analyse if this singularity is hidden behind a horizon. The horizons correspond to the locations where $V(r) = 0$. Though we are unable to solve this equation analytically, we first plot, in figure 1, the function $V(r)$ for some different values of m and for $n = 4$. In this figure, the other parameters such as l, β are kept fixed. First of all, let us note that there can be one or two horizons depending on the value of m . Furthermore, for certain choices of m there can be no horizon, leading to a naked singularity at the origin. To have further understanding

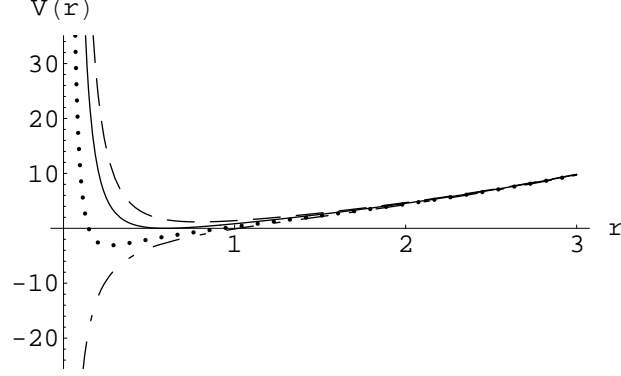


Figure 1: The metric function $V(r)$ as a function of r for $n = 4, \beta = 1, q = 1, l = 1$. The dashed line, solid line, dotted line and dash-dotted line are for $m = 1.5, 2.062, 2.6, 3$ respectively.

on the nature of the horizons, we plot in figure 2, the mass as a function of the horizon radius. The mass parameter of the hole can be expressed in terms of horizon radius (r_+) as

$$m = r_+^{n-2} + \left[\frac{4\beta^2}{n(n-1)} + \frac{1}{l^2} \right] r_+^n - \frac{2\sqrt{2}\beta r_+}{n(n-1)} \sqrt{2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2} + \frac{2(n-1)q^2}{nr_+^{n-2}} {}_2F_1 \left[\frac{n-2}{2n-2}, \frac{1}{2}, \frac{3n-4}{2n-2}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r_+^{2n-2}} \right]. \quad (14)$$

In figure 2, $m(r_+)$ is shown for different values of β . For a given β , the number of horizons depend clearly upon the choice of m . If we focus our attention to the solid line ($\beta = 5$), we see that upto certain m , there are two horizons. As we decrease the mass further, the two horizons meet. We

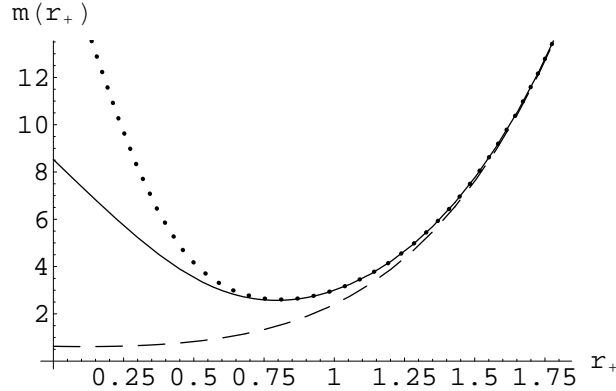


Figure 2: The mass m as a function of r_+ for adS Born-Infeld holes for $n = 4, l = 1, q = 1, \beta = 16$ (dotted line) ,5 (solid line) ,0.1 (dashed line).

then call the black hole extremal. It is easy to find out from $V(r)$ that this happens when r_+ satisfies the following condition:

$$(n-2)r_+^{n-3} + \left[\frac{4\beta^2}{n-1} + \frac{n}{l^2} \right] r_+^{n-1} - \frac{2\sqrt{2}\beta}{n-1} \sqrt{2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2} = 0. \quad (15)$$

For $\beta = 0.1$ there is only a single horizon, as equation (15) does not have a real solution for r_+ and hence extremality condition can not be satisfied. As we will see later that when this condition is

satisfied, the temperature of the black hole vanishes. We also notice that the behaviour of $V(r)$ crucially depends on the ratio m/A , where A is defined in (13). If $m/A \geq 1$, $V(r)$ behaves like that of a Schwarzschild black hole close to the origin. On the other hand, for $m/A < 1$, behaviour of $V(r)$ is more like the Reissner-Nordstrom one.

From equation (7), we can also calculate the gauge field associated with this configuration. It is given by

$$A_t = \frac{1}{c} \frac{q}{r^{n-2}} {}_2F_1 \left[\frac{n-2}{2n-2}, \frac{1}{2}, \frac{3n-4}{2n-2}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r^{2n-2}} \right] - \Phi, \quad (16)$$

where q is related to the black hole charge Q via

$$Q = 2\sqrt{2(n-1)(n-2)\omega_{n-1}q}.$$

In equation (16), Φ is the gauge potential. We will choose Φ in such a way that A_t is zero at the horizon. This gives

$$\Phi = \frac{1}{c} \frac{q}{r_+^{n-2}} {}_2F_1 \left[\frac{n-2}{2n-2}, \frac{1}{2}, \frac{3n-4}{2n-2}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r_+^{2n-2}} \right]. \quad (17)$$

Behaviour of Φ as a function of the horizon size for $n = 4, q = 4, l = \beta = 1$ is shown in figure 3. Notice that Φ is finite even when $r_+ = 0$.

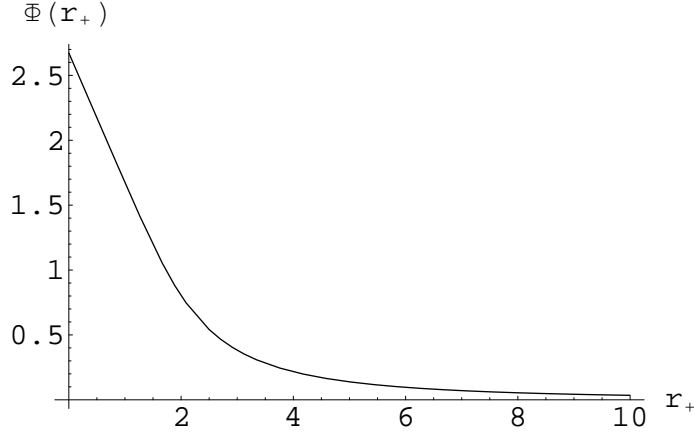


Figure 3: The potential Φ as a function of r_+ for $n = 4, l = 1, \beta = 1, q = 4$. Note that Φ is finite for $r_+ = 0$.

We would now like to make some brief comments on the Born-Infeld deSitter black holes. These are the solutions in the presence of a positive cosmological constant. This can be found from the earlier expression of $V(r)$ by replacing l^2 by $-l^2$. More explicitly,

$$\begin{aligned} V(r) &= 1 - \frac{m}{r^{n-2}} + \left[\frac{4\beta^2}{n(n-1)} - \frac{1}{l^2} \right] r^2 - \frac{2\sqrt{2}\beta}{n(n-1)r^{n-3}} \sqrt{2\beta^2 r^{2n-2} + (n-1)(n-2)q^2} \\ &+ \frac{2(n-1)q^2}{nr^{2n-4}} {}_2F_1 \left[\frac{n-2}{2n-2}, \frac{1}{2}, \frac{3n-4}{2n-2}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r^{2n-2}} \right]. \end{aligned} \quad (18)$$

This metric, in the asymptotic region, goes to the Reissner-Nordstrom deSitter black holes with a β dependent Born-Infeld correction. While near $r = 0$, $V(r)$ behaves similar to equation (12) with $l^2 \rightarrow -l^2$. Therefore, the singularity structure of this solution is the same as the previous one. Now, turning our attention to the nature of the horizon, we find that it is best described in terms of a plot of m as a function of r_+ . This is shown in figure 4, where we have given a plot of $m(r_+)$ for different l , keeping the other parameters fixed. For small l , m monotonically decreases with r_+ . So, the configuration has only one horizon (inner or cosmological depending on the mass). However, for large l , the solution has three horizons; out of them the largest one is the cosmological horizon and the other two are the black hole inner horizon and the event horizon. For fixed l , if we decrease m , these two horizons of the black hole come closer and they meet at a certain value of m . For even lower values of m , only the cosmological horizon exists. Now for even larger value of l (shown in dashed line in the figure), there can be only two horizons. Out of that, the larger one is the cosmological horizon. As we now increase m , the event and the cosmological horizons meet. Beyond that value of m we get naked singularity (all horizons vanish) at the origin.

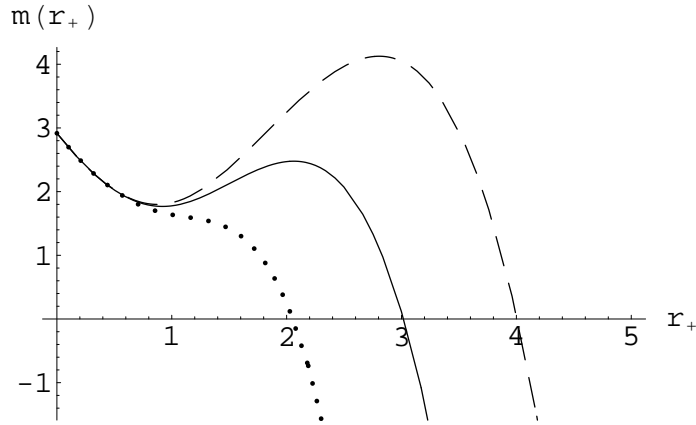


Figure 4: Mass m as a function of r_+ for Born-Infeld deSitter holes for $n = 4$, $\beta=1$, $q = 1$, $l = 2$ (dotted line), 3 (solid line) , 4 (dashed line).

3 Thermodynamics

We now would like to study the thermodynamical properties of the black holes we have just found.

The Hawking temperature of the hole can be calculated using the relation

$$T = \frac{\kappa}{2\pi}, \quad (19)$$

where κ is the surface gravity and is given by

$$\kappa = -\frac{1}{2} \frac{dg_{tt}}{dr} \Big|_{r=r_+}. \quad (20)$$

κ can be calculated explicitly from the metric functions. The temperature is then,

$$T = \frac{1}{4\pi} \left[\frac{n-2}{r_+} + \left\{ \frac{4\beta^2}{n-1} + \frac{n}{l^2} \right\} r_+ - \frac{2\sqrt{2}\beta}{(n-1)r_+^{n-2}} \sqrt{2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2} \right]. \quad (21)$$

We note that when r_+ is such that the right hand side of the above equation is zero, the temperature of the black hole is zero. We notice that this gives the same constraint on r_+ that we have already encountered while discussing extremal black hole (see eq (15)). From here we conclude, in the extremal limit, the temperature of the black hole is zero.

Using the standard formula for entropy

$$S = \int T^{-1} \left(\frac{\partial M}{\partial r_+} \right)_Q dr_+, \quad (22)$$

we get

$$S = 4\pi\omega_{n-1}r_+^{n-1} \quad (23)$$

as the entropy of the black hole. It is indeed proportional to the area of the horizon. Now that we have all the relevant thermodynamic quantities, we can easily verify that the first law of thermodynamics. We find that

$$dM = TdS + \Phi dQ, \quad (24)$$

is satisfied. To find the stability of the black hole, it is important to find the specific heat of the hole. This can be easily evaluated using

$$C_Q = \left(\frac{\partial M}{\partial T} \right)_Q = \left(\frac{\partial M}{\partial r_+} \right)_Q / \left(\frac{\partial T}{\partial r_+} \right)_Q, \quad (25)$$

where M and T are given in eqs. (11) and (21). In figure 5, we have plotted C_Q as a

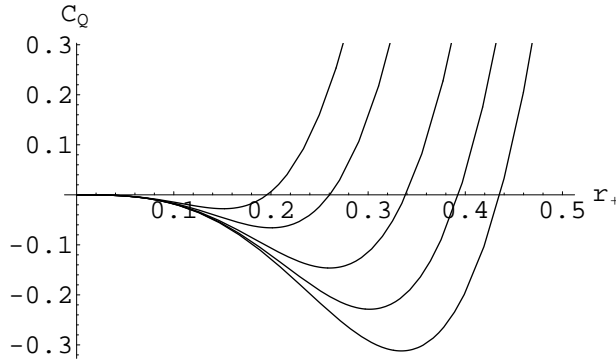


Figure 5: Specific heat at fixed charge as a function of r_+ where $n = 4, l = .2, \beta = 1$ and $q = .5, 1, 2, 3, 4$ from left to right respectively.

function of horizon radius r_+ . We see from the figure that the specific heat is negative if r_+ is less than certain value; making the hole *unstable*. We therefore conclude that the large Born-Infeld black holes with adS asymptotics are stable against fluctuations. We can now write down the

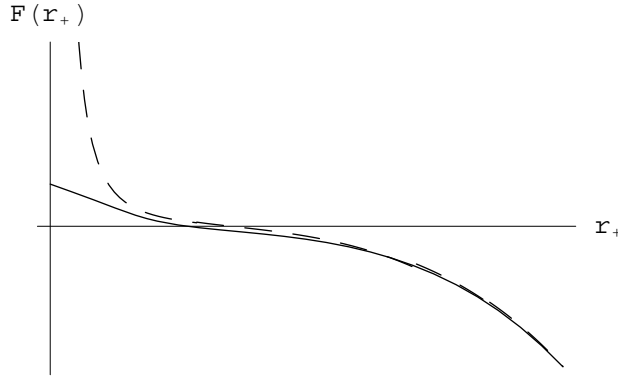


Figure 6: Free energy at fixed charge for Born-Infeld (solid line) and RNAdS (dashed line) black holes for $n = 4$, $l = 1$, $\beta = 10$ and $q = 10$.

thermodynamics canonical potential at *fixed* charge Q , $F = E - TS$, where $E = M - M_e$ and M_e is the ADM mass of the extremal black hole that follows from the condition (15). We get

$$\begin{aligned}
F = & \omega_{n-1} \left[r_+^{n-2} - \left\{ \frac{4\beta^2}{n(n-1)} + \frac{1}{l^2} \right\} r_+^n + \frac{2\sqrt{2}\beta r_+}{n(n-1)} \sqrt{2\beta^2 r_+^{2n-2} + (n-1)(n-2)q^2} \right. \\
& \left. + \frac{2(n-1)^2 q^2}{nr_+^{n-2}} {}_2F_1 \left[\frac{n-2}{2n-2}, \frac{1}{2}, \frac{3n-4}{2n-2}, -\frac{(n-1)(n-2)q^2}{2\beta^2 r_+^{2n-2}} \right] - M_e \right]. \quad (26)
\end{aligned}$$

In figure 6, we have shown F as a function of r_+ for Born-Infeld black holes and for the standard RNAdS black holes. This figure also shows that the larger black holes are more stable than the smaller ones. We also see that for large r_+ , F behaves similar to that of RNAdS black holes. However, for small holes, their behaviours are distinctly different.

4 Discussion

In this letter, we have constructed charged adS and dS black holes of Einstein-Born-Infeld actions. We believe that this is only the first step before we analyse, following the adS/CFT correspondence, the behaviour of the boundary theory caused by higher order gauge field perturbations in the bulk. Beside the motivation coming from adS/CFT side, Born-Infeld lagrangian appears very frequently in string theory. So we expect that it is worthwhile to know various properties of black hole solutions in this theory.

Though, in this paper we have constructed the Born-Infeld black holes in the presence of a cosmological constant and discussed their thermodynamical properties, many issues however still remain to be investigated. We know that Reissner-Nordstrom adS black holes undergo Hawking-Page phase transition. This transition gets modified as we include Born-Infeld corrections into account. We hope to carry out a detail study on this issue in the future. Furthermore, in the context of brane world cosmology, it was found that a brane moving in a Reissner-Nordstrom adS background generates non-singular cosmology [14]. However, as shown in [15], the brane always crosses the inner horizon of the bulk geometry, creating instability. It would be interesting to

study cosmology on the brane when it is moving in the charged black hole backgrounds that we have constructed. Note that since these charged holes does not have inner horizon for certain range of parameters, we may generate non-singular cosmology without creating the instabilities that we have just mentioned.

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